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# THE CELLULAR STRUCTURE OF FORMAL HOMOTOPY TYPES

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## Introduction

This paper is an attempt to give a new characterization of formal spaces, in terms of their cell-structure. The formal spaces, as introduced by Sullivan [7], are defined by the property that their rational cohomology algebra determines the rational homotopy type. A formal space is a most appropriate one for doing rational homotopy computations, but the problem of deciding the formality is known to be sufficiently complicated.

We propose a characterization of the formality of a complex, in terms of the attaching maps, in Theorem 3.4, which is the main result of Section 3. It complements other known purely algebraic characterizations of formality (see [2] and [5]).

The results in Section 3 depend on the technique of constructing d.g.a. minimal models which contain information about a given cell decomposition (traditionally, the construction follows the pattern of the Postnikov decomposition - see Section 1). This technique is developed in Section 2, whose main result is Proposition 2.3.

In Section 4, we introduce a class of rational homotopy types, which admit cell decompositions of a simple kind (see Definitions 4.1, 4.4) and give a characterization of this property, in Theorem 4.5. In Corollary 4.7, we indicate a class of examples of such spaces, whose formality will be investigated in another paper, in the context of rational Poincaré duality spaces.

## 1. Preliminaries

We recall some of the main results of Sullivan [7] (see also [1], [2], [6]). Concerning notations and terminology, we will follow those in [2], with minor changes.

A *minimal* differential graded commutative algebra (d.g.a.) $\mathcal{M}$ is required to be freely generated as a graded commutative algebra (g.a.), with generators, say $V=\bigoplus_{p>1} V^p$, that is $\mathcal{M}=\Lambda(V)$ (see also [2, §4]) with a decomposable differential, which we will write

$$d(V)\subset D\mathcal{M}=\mathcal{M}^+\cdot\mathcal{M}^+$$

(compare with [2, §1]). Sullivan shows that any (simply connected) d.g.a. $\mathcal{A}$ has a *minimal model*, i.e., a minimal algebra $\mathcal{M}$, together with a d.g.a. map inducing isomorphism in cohomology, $\varrho:\mathcal{M}\to\mathcal{A}$, unique up to isomorphism (see [2, Theorem 1.1]). It will be important for us to have a precise description of the inductive construction of the model, which is: given any $(n-1)$-stage model, $\varrho_{n-1}:\mathcal{M}_{n-1}\to\mathcal{A}$, an $n$-stage model is obtained by an elementary extension

$$\mathcal{M}_n=\mathcal{M}_{n-1}\otimes_d\Lambda_n(V^n),$$

where

$$V^n=C^n\oplus K^n,\qquad C^n=\operatorname{coker} H^n\varrho_{n-1},\qquad K^n=\operatorname{Ker} H^{n+1}\varrho_{n-1},$$

$$d\,|\,C^n=0, \tag{1.1}$$

and, passing to cohomology $[d\,|\,K^n]=$ inclusion (see Example 4.9).

At this point, we introduce the following:

**1.2. Definition.** A minimal $\mathcal{M}$ is said to be *in normal form* if

$$\operatorname{Ker}[d\,|\,V]=\operatorname{Ker}(d\,|\,V).$$

(The right hand side will also be denoted by $C_{\mathcal{M}}$). This assumption is made in order to avoid unnecessary repetitions in the d.g.a. structure, as in

$$\mathcal{M}=\Lambda_2(x)\otimes\Lambda_3(y)\otimes\Lambda_5(z)\quad\text{with } dx=0,\ dy=x^2,\ dz=x^3,$$

which can be put in the normal form

$$\mathcal{M}'=\Lambda_2(x')\otimes\Lambda_3(y')\otimes\Lambda_5(z')\quad\text{with } dx'=0,\ dy'=x'^2,\ dz'=0.$$

Actually, the situation is general: the models constructed by the algorithm (1.1) will always be in normal form which will be for us the prefered one, within its isomorphism class. With the normal form hypothesis, one can easily see that the following useful relations hold:

$$H^p\mathcal{M}=H^p\mathcal{M}_p=C^p\oplus H^p\mathcal{M}_{p-1}. \tag{1.3}$$

Sullivan also introduces a notion of homotopy in the d.g.a. category, and then shows how to construct the model of a d.g.a. morphism $f:\mathcal{A}\to\mathcal{A}'$; namely, given models $\varrho:\mathcal{M}\to\mathcal{A}$, $\varrho':\mathcal{M}'\to\mathcal{A}'$, there exists a (unique up to homotopy) model for $f$, $\hat{f}:\mathcal{M}\to\mathcal{M}'$, which satisfies $f\varrho\simeq\varrho'\hat{f}$ [2, Corollary 1.5]. We make the observation that it is not hard to see that the existence and uniqueness assertions also hold for $p$-stage models, which will be denoted by $\varrho_p:\mathcal{M}_p\to\mathcal{A}$, and $\hat{f}_p:\mathcal{M}_p\to\mathcal{M}'_p$.

The geometric meaning of the minimal model theory is revealed by the Sullivan–Whitney–Thom construction of the *de Rham* d.g.a. $\mathscr{E}_X$ of a space $X$, which gives a functorial link between the (homotopy) topological and algebraic categories, via the minimal model construction: $X \to \mathscr{M}_X$ (model of $\mathscr{E}_X$), $f \to \hat{f}$ (model of $\mathscr{E}_f$) (see [2, §2]).

Finally Sullivan shows that the minimal model $\mathscr{M}_X$ uniquely determines the *rational homotopy type* of $X$, i.e. the homotopy type of the *localization* $X_0$ (see [6, I.4]).

This fact is essentially proved by showing that the structure of $\mathscr{M}_X$ exactly mirrors the Postnikov decomposition of $X_0$ [2, §3]. When trying to construct models which reflect the cell structure of a finite complex $X$, we begin by noting that $\dim_Q H^*X < \infty$, and then try to exploit this simple observation. If $\mathscr{A}$ is a d.g.a., the notation c-dim $\mathscr{A} \leq n$ will mean that $H^p\mathscr{A} = 0$ for $p > n$.

**1.4. Definition.** Given an $(n-1)$-stage minimal $\mathscr{M}_{n-1}$ and a $Q$-vector space $C$, the *canonical extension* of $\mathscr{M}_{n-1}$ by $C$, with c-dim $\leq n$ will be the minimal algebra $\mathscr{M}$, constructed by the following sequence of elementary extensions:

$$\mathscr{M}_n = \mathscr{M}_{n-1} \otimes_d \Lambda_n(C) \otimes \Lambda_n(H^{n+1}\mathscr{M}_{n-1}) \quad \text{with } d \,|\, C = 0,\ [d \,|\, H^{n+1}\mathscr{M}_{n-1}] = \text{id},$$

$$\mathscr{M}_p = \mathscr{M}_{p-1} \otimes_d \Lambda_p(H^{p+1}\mathscr{M}_{p-1}) \quad \text{with } [d] = \text{id}, \quad \text{for } p > n.$$

Our first preliminary remark is given by the following lemma and is a direct consequence of the construction (1.1):

**1.5. Lemma.** *If* c-dim $\mathscr{M} \leq n$, *then* $\mathscr{M}$ *is isomorphic to a canonical extension of* $\mathscr{M}_{n-1}$ *with* c-dim $\leq n$.

When c-dim $\mathscr{M} < \infty$, not only $\mathscr{M}$ is determined by a finite-stage subalgebra, but the formality of $\mathscr{M}$ can also be decided at the finite-stage level. Sullivan introduced the notion of *formal* minimal algebra $\mathscr{M}$ be requiring the existence of a d.g.a. map $\varrho : \mathscr{M} \to H^*\mathscr{M}$, inducing the identity in cohomology (see [2, §4]). It is easy to see what becomes of this condition when the c-finiteness hypothesis is added, namely:

**1.6. Lemma.** *Suppose* c-dim $\mathscr{M} \leq n$. *Then* $\mathscr{M}$ *is formal iff there exists* $\varrho_{n-1} : \mathscr{M}_{n-1} \to H^*\mathscr{M}_{n-1}$, *such that* $H^p\varrho_{n-1} = \text{id}$ *for* $p \leq n$.

The following formulation of a well-known first obstruction to formality will also be useful:

**1.7. Definition.** Let $\mathscr{M}$ be a minimal algebra in normal form. We will say that $H^*\mathscr{M}$ is *spherically generated* if the cohomology classes $[C_{\mathscr{M}}]$ generate $H^*\mathscr{M}$ as an algebra. As usual, the definition may be extended to arbitrary d.g.a.'s and spaces, using minimal models in normal form (for the geometric meaning and for another interpretation, see [5, §8]).

**1.8. Lemma.** *Let $\mathcal{M}$ be minimal in normal form. If $\mathcal{M}$ is formal, then $H^*\mathcal{M}$ is spherically generated.*

The statement is to be compared with [5, Theorem 8.12], and the proof is achieved with an argument like that in [2, Theorem 4.1]. We point out that the hypothesis of normal form is necessary in this formulation, as shown by the example already used:

$$\mathcal{M}=\Lambda_2(x)\otimes\Lambda_3(y)\otimes\Lambda_5(z),$$

where the class $[z-xy]\in H^5\mathcal{M}$ cannot be expressed as a polinomial in $[x]$.

## 2. The minimal model of a cellular decomposition

We begin by describing the construction of the model of a space $X$, obtained from a space $A$ by attaching $(n+1)$-cells, in terms of the model of $A$ and of the models of the attaching maps.

Let $A$ be a space, cohomologically simply connected (i.e., $H^1(A;Q)=0$) and whose rational homology is of finite type. Let us denote by $j:A\hookrightarrow X$, the result of attaching $(n+1)$-cells, with a finite number of attaching maps $\{f_\lambda:S^n\to A\}$.

We choose the following initial data for our construction: $n$-stage models $\gamma_n:\mathcal{S}_n\to\mathcal{E}_{S^n}$, $\alpha_n:\mathcal{A}_n\to\mathcal{E}_A$, and $\hat{f}_{\lambda,n}:\mathcal{A}_n\to\mathcal{S}_n$.

The construction goes as follows: we denote by $L_\lambda$ the linear map

$$\pi^n\mathcal{A}_n\xrightarrow{\pi^n\hat{f}_{\lambda,n}}\pi^n\mathcal{S}_n,$$

we write

$$\mathcal{A}_n=\mathcal{A}_{n-1}\otimes_{d_A}\Lambda_n(W^n),$$

thus identifying $L_\lambda$ with an element in the $Q$-dual $W^{n*}$, and we denote by $V^n$ the subspace defined by $V^n=\bigcap_\lambda \operatorname{Ker} L_\lambda$. We then define a sub d.g.a. of $\mathcal{A}_n$, $n$-minimal, by

$$\mathcal{X}_n=\mathcal{A}_{n-1}\otimes_{d_X}\Lambda_n(V^n),\quad \text{with } d_X=d_{A|V^n}. \tag{2.1}$$

**2.2. Lemma.** *The inclusion $i:\mathcal{X}_n\hookrightarrow\mathcal{A}_n$ represents an $n$-stage model of the map $j$.*

**Proof.** We first extend the given $n$-stage models to minimal models $\mathcal{S}$, $\mathcal{A}$, and $\hat{f}_\lambda$. The rational cohomology properties of $j$ assure the existence of a d.g.a. map $\varphi_{n-1}:\mathcal{A}_{n-1}\to\mathcal{E}_X$, with the property $\mathcal{E}_j\circ\varphi_{n-1}\simeq\alpha_{n-1}$ (compare [2, Theorem 1.2]), which we then extend to a model: $\varphi':\mathcal{X}'\to\mathcal{E}_X$ (so $\mathcal{X}'_{n-1}=\mathcal{A}_{n-1}$), and so a model $\hat{j}$ can be constructed with the property that, when restricted to $\mathcal{X}'_{n-1}$, it equals the identity. We first show $\operatorname{Im}\pi^n\hat{j}=V^n$.

In order to prove the nontrivial inclusion, namely that of $V^n$, we start with $x\in V^n$ and consider the d.g.a. inclusion

$$\mathscr{X} = \mathscr{A}_{n-1} \otimes_{d_A} \Lambda_n(x) \overset{\hat{g}}{\hookrightarrow} \mathscr{A},$$

which, by Sullivan, can be realized as the model of a map between rational spaces $g: A_0 \to Z_0$. Recalling the definition of $V^n$ and using again the Sullivan equivalence of categories we see that $g \circ f_{\lambda,0} \simeq$ point, for any $\lambda$, at the level of localizations. Denoting by $j_0: A_0 \hookrightarrow X_0$ the inclusion into the adjunction space obtained from $A_0$ by attaching cones over localized $n$-spheres, which just localizes $j$, we deduce the existence of an extension $h$ of $g$ over $X_0$: $hj_0 = g$. Passing back to minimal models, we have: $x = \hat{g}(x) = \hat{j}\hat{h}(x)$, so indeed: $x \in \operatorname{Im} \pi^n \hat{j}$.

Due to the cohomological properties of $j$, the minimal model version of the relative Hurewicz isomorphism theorem shows that $\pi^p \hat{j}_n$ is an isomorphism for $p < n$, and $\pi^n \hat{j}_n$ is an isomorphism onto $V^n$. In other words, we have a d.g.a. isomorphism $\hat{j}_n: \mathscr{X}'_n \overset{\sim}{\to} \mathscr{X}_n$. The lemma follows.

Introducing a c-finiteness assumption on $A$, we can make more precise the conclusion of the lemma:

**2.3. Proposition.** *Suppose, in addition to the conditions in Lemma* 2.2, *that* c-dim $A \le n$, *and* $\mathscr{A}_n$ *is in normal form. Then the model* $\mathscr{M}_X$ *is given by the canonical extension of* $\mathscr{X}_n$ *(as constructed in* (2.1)) *by* $C_X^{n+1}$, *with* c-dim$\le n+1$, *where* $\dim C_X^{n+1} = r - \operatorname{cod} V^n$ *(r stands for the number of* $(n+1)$*-cells and* cod $V^n$ *is the codimension of* $V^n$ *in* $W^n$*).*

**Proof.** Since, clearly, c-dim $X \le n+1$, Lemma 1.5 is available. We only have to check the dimension of $C_X^{n+1}$. Relation (1.3) gives

$$\dim C_X^{n+1} = \dim H^{n+1}X - \dim H^{n+1}\mathscr{X}_n,$$

and the cohomology exact sequence of the pair $(X, A)$ gives

$$\dim H^{n+1}X = r - (\dim H^n A - \dim H^n X),$$

which equals $r - (\dim H^n \mathscr{A}_n - \dim H^n \mathscr{X}_n)$. Examining the exact sequences produced by elementary extension (compare [6, Lemma II.7])

$$0 \to H^n \mathscr{A}_{n-1} \to H^n \mathscr{A}_n \to W^n \to H^{n+1}\mathscr{A}_{n-1} \to H^{n+1}\mathscr{A}_n \to 0$$

and

$$0 \to H^n \mathscr{A}_{n-1} \to H^n \mathscr{X}_n \to V^n \to H^{n+1}\mathscr{A}_{n-1} \to H^{n+1}\mathscr{X}_n \to 0$$

we find that

$$\dim H^{n+1}\mathscr{X}_n = (\dim W^n - \dim V^n) - (\dim H^n \mathscr{A}_n - \dim H^n \mathscr{X}_n)$$

which finally gives $\dim C_X^{n+1} = r - \operatorname{cod} V^n$.

Proposition 2.3 gives the inductive step in the construction of the minimal model of a cell decomposition, which we will restate, for further notational use.

Let $X$ be a CW-complex, cohomologically simply connected and of finite type.

Suppose we are given $\mathscr{X}_{n,n}$, an $n$-stage model in normal form of the $n$-skeleton $X^n$:

$$\mathscr{X}_{n,n} = \Lambda_2(W_n^2)\otimes\cdots\otimes\Lambda_n(W_n^n).$$

We will also write $C_{\mathscr{X}_n} = \bigoplus C_n^p$, where $\mathscr{X}_n$ is a minimal model, in normal form, which extends $\mathscr{X}_{n,n}$. In order to pass to the $(n+1)$-skeleton, we make the construction described in (2.1):

$$\mathscr{X}_{n+1,n} = \mathscr{X}_{n,n-1}\otimes_d \Lambda_n(W_{n+1}^n),$$

where $W_{n+1}^n$ was previously called $V^n$; this will provide an $n$-stage model for $X^{n+1}$, still in normal form. We now consider the vector space $C_{n+1}^{n+1}$, given by

$$\dim C_{n+1}^{n+1} = r_{n+1} - \operatorname{cod} W_{n+1}^n.$$

We then define

$$W_{n+1}^{n+1} = C_{n+1}^{n+1} \oplus H^{n+2}\mathscr{X}_{n+1,n},$$

observing that we have, by construction $W_{n+1}^p = W_n^p$ for $p<n$, and we finally put:

$$\mathscr{X}_{n+1,n+1} = \mathscr{X}_{n+1,n}\otimes_d \Lambda_{n+1}(W_{n+1}^{n+1})$$
$$\text{where } d\,|\,C_{n+1}^{n+1} = 0 \quad\text{and}\quad [d\,|\,H^{n+2}\mathscr{X}_{n+1,n}] = \mathrm{id}. \tag{2.4}$$

**2.5. Corollary.** *$\mathscr{X}_{n+1,n+1}$ represents an $(n+1)$-stage model in normal for $X^{n+1}$. $\mathscr{X}_{n+1,n}$ is an n-stage model for X. If X is $(n+1)$-dimensional, then the model of X is the canonical extension of $\mathscr{X}_{n+1,n}$, by $C_{n+1}^{n+1}$, with* c-dim$\le n+1$.

In order to complete the statement in Proposition 2.3, we will show that, given $A$, any algebraic data ($r$ and $V^n$) can be realized by attaching $(n+1)$-cells to $A$:

**2.6. Proposition.** *Let A be simply connected and with finite type rational homology. Let $\mathscr{S}_n$ and $\mathscr{A}_n$ be fixed n-stage models for $S^n$ and A. If $V^n$ is a subspace of $\pi^n\mathscr{A}_n$ and r is a number, $r\ge \operatorname{cod} V^n$, then there exist r attaching maps $f_\lambda : S^n\to A$ such that $V^n = \bigcap_\lambda \operatorname{Ker} \pi^n \hat{f}_{\lambda,n}$.*

**Proof.** Since $r\ge\operatorname{cod} V^n$, there exist $r$ elements in $W^{n*}$, say $\{L_\lambda\}$, such that $V^n = \bigcap \operatorname{Ker} L_\lambda$. We show now that, for a such given $L$, there is a map, $f: S^n\to A$, such that $\operatorname{Ker} L = \operatorname{Ker} \pi^n \hat{f}_n$. We will use extensions of the given $n$-stage models, denoted by $\mathscr{S}$ and $\mathscr{A}$, and a localization $l: A\to A_0$. We note that, by simply connectivity, we have an isomorphism

$$\pi_n(A)\otimes Q \xrightarrow{l_\#\otimes \mathrm{id}} \pi_n(A_0)\otimes Q = \pi_n(A_0)$$

(see [6, Theorem I.4]). The given $L$ appears as $\pi^n\hat{g}_0$, for some d.g.a. map $\hat{g}_0 : \mathscr{A}\to\mathscr{S}$, minimal model of $g_0 : S_0^n \to A_0$. This in turn gives rise to a map $f_0 : S^n\to A_0$, with the property $L = \pi^n \hat{f}_0$. At the homotopy groups level, we find a map $f$ and a nonzero integer $q$ such that $l_\#([f]) = q\cdot[f_0]$, which implies that $lf \simeq f_0 f_q$ where

$f_q: S^n \to S^n$ is a rational homotopy equivalence. Passing to minimal models, we have $\operatorname{Ker} \pi^n \hat{f}_0 = \operatorname{Ker} \pi^n \hat{f}$, which finishes the proof.

As a first application of our construction, we give a rapid proof of the integral realization of a minimal algebra in the finite c-dim case (see Theorem 10.2(ii) from [7]).

**2.7. Proposition.** *Let $\mathcal{M}$ be minimal of finite type, and suppose* c-dim $\mathcal{M} \leq n$. *Then there exists a finite simply connected complex $X$, with* $\dim X \leq n$, *such that* $\mathcal{M}_X = \mathcal{M}$.

**Proof.** The proof goes by induction. When passing to $n+1$, let us note that $\mathcal{M}$ is isomorphic to the canonical extension of $\mathcal{M}_n$ by $C_{\mathcal{M}}^{n+1}$, with c-dim $\leq n+1$. We note $\mathcal{A}_{n-1} = \mathcal{M}_{n-1}$, and then define $\mathcal{A}$ as the canonical extension of $\mathcal{A}_{n-1}$ by $C_{\mathcal{M}}^n$, with c-dim $\leq n$. Let $A$ realize $\mathcal{A}$. We recall $W^n = C^n \oplus H^{n+1}\mathcal{M}_{n-1}$. We may well suppose $\mathcal{M}$ in normal form and write $\pi^n \mathcal{M}_n = C^n \oplus K$ with $[d \mid K]$ an isomorphism onto its image. We define

$$V^n = C^n \oplus \operatorname{Im}[d] \quad \text{and} \quad r = \operatorname{cod} V^n + \dim C_{\mathcal{M}}^{n+1}.$$

Using 2.6, we realize $V^n$ and $r$, and obtain a finite complex $X$. 2.3 tells us that $\mathcal{M}_X$ is isomorphic to the canonical extension of $\mathcal{X}_n$ by $C_X^{n+1}$, with c-dim $\leq n+1$, where $\dim C_X^{n+1} = r - \operatorname{cod} V^n = \dim C_{\mathcal{M}}^{n+1}$. A little analysis of the construction of $\mathcal{A}_n$ and $\mathcal{X}_n$ shows $\mathcal{X}_n$ to be isomorphic to $\mathcal{M}_n$, so indeed $\mathcal{M}_X = \mathcal{M}$.

**2.8. Remark.** Let us say that a complex $X$ is *rationally structured* if the inclusion of any skeleton, $X^m \hookrightarrow X^{m+1}$, induces rational homology isomorphism in all dimensions $p \leq m$. With this definition let us note that our proof gives a little more, namely: supposing, inductively, the existence of a rationally structured realization, we can easily check the condition for $X$, when $m = n$, $p = n$, by simply estimating with relations (1.3):

$$\dim H^n A = \dim H^n \mathcal{A}_n = \dim C^n + \dim H^n \mathcal{A}_{n-1},$$

$$\dim H^n X = \dim H^n \mathcal{X}_n = \dim C^n + \dim H^n \mathcal{A}_{n-1}.$$

**2.9. Remark.** The condition of being rationally structured seems to be adequate for rational homotopy purposes, since it guarantees that the rational homotopy type of skeleta are rational homotopy type invariants of the complex, and even the number of cells in each dimension, facts which are definitely false for arbitrary complexes, as shown by simple examples.

## 3. The formality test for cellular structures

Our starting point will be the following lemma, which slightly improves a result of Sullivan (see the examples of formal spaces in [7, §12]):

**3.1. Lemma.** *Let $A$ be cohomologically simply connected, and suppose* c-dim $A \leq n+1$. *Given a map $j: A \to X$, which induces rational cohomology isomorphism in dimensions $p<n$ and a monomorphism in dimension $p=n$, if $X$ is formal, then $A$ is formal.*

**Proof.** Let $\varphi: \mathcal{M} \to \mathcal{E}_X$ be a model. Since $X$ is formal, there exists $\psi: \mathcal{M} \to H^*X$ such that $\psi^* = \varphi^*$. Restricting these models to $\mathcal{M}_{n-1}$ and composing with the obvious maps induced by $j$, we obtain $\varphi_{n-1}: \mathcal{M}_{n-1} \to \mathcal{E}_A$, an $(n-1)$-stage model, and $\psi_{n-1}: \mathcal{M}_{n-1} \to H^*A$, with the property $\psi^*_{n-1} = \varphi^*_{n-1}$. When extending these $(n-1)$-stage models, using (1.1) we observe that

$$\dim C^n = \dim H^n A - \dim H^n \mathcal{M}_{n-1}$$

and

$$K^n = \operatorname{Ker} H^{n+1}\varphi_{n-1} = \operatorname{Ker} H^{n+1}\psi_{n-1},$$

so we have a common $n$-stage model for $\mathcal{E}_A$ and $H^*A$, say $\mathcal{M}'_n$. Given a common $p$-stage model, $\mathcal{M}'_p$, $p \geq n$, the same argument shows that, due to the hypothesis of c-dim $A \leq n+1$, it can be extended to a common $\mathcal{M}'_{p+1}$.

**3.2. Corollary.** *The connected sum of two cohomologically simply connected manifolds which are formal spaces, is again formal.*

**Proof.** Let $M$ and $M'$ be $(n+1)$-dimensional. Their connected sum, $M \# M'$, with a $(n+1)$-cell attached, has the homotopy type of the connected sum (with base points) $M \vee M'$, which is known to be formal. We can now apply Lemma 3.1.

We go back to the situation described in Proposition 2.3, and add a few notations: $\langle d_A \rangle$ will stand for the composed map

$$W^n \xrightarrow{[d_A]} H^{n+1}\mathcal{A}_{n-1} \to H^{n+1}\mathcal{A}_{n-1}/DH^{n+1}\mathcal{A}_{n-1}$$

(where $D$ denotes the decomposable elements), and $\langle d_X \rangle$ will stand for the restriction $\langle d_A \rangle \mid V^n$. We first check the obstruction to formality described in Lemma 1.8.

**3.3. Lemma.** *Suppose further that $H^*A$ is spherically generated. Then $H^*X$ is spherically generated iff $\langle d_X \rangle$ is onto.*

**Proof.** Extending the $n$-stage model in normal form $\mathcal{X}_n$ produced by Lemma 2.2, to a model of $X$, say $\mathcal{X}$, still in normal form, it is easy to see, using (1.3), that the hypothesis on $X$ implies $H^{n+1}\mathcal{X}_n = DH^{n+1}\mathcal{X}_n$. Examining the exact sequence

$$V^n \xrightarrow{[d_X]} H^{n+1}\mathcal{A}_{n-1} \to H^{n+1}\mathcal{X}_n \to 0,$$

we can immediately conclude that $\langle d_X \rangle$ is onto. Conversely, the surjectivity of $\langle d_X \rangle$ implies that $H^{n+1}\mathcal{X}_n = DH^{n+1}\mathcal{X}_n$. We note that, due to the c-dim assumption, it will be sufficient to show that $H^p\mathscr{X}_n$ is generated by $[C_{\mathscr{X}_n}]$, for $p \le n+1$. $H^*A$ being spherically generated, $H^p\mathscr{A}_n$ will be generated by $[C_{\mathscr{A}_n}]$, for $p \le n$. Recalling the construction of $\mathscr{X}_n$ and using again (1.3), this implies the desired assertion for $\mathscr{X}_n$, for $p \le n$. The elements in $H^{n+1}\mathscr{X}_n$ being decomposable, $H^*X$ results spherically generated.

We are ready to prove the main result of this section, which gives the characterization of the formality of a space $X$, obtained by attaching cells to $A$, in terms of $A$ and of the attaching maps.

**3.4. Theorem.** *Let $A$ be cohomologically simply connected, with finite type rational homology, and with* c-dim $A \le n$, $n > 1$. *Let $X$ be obtained from $A$, by finitely attaching $(n+1)$-cells.*

*Start with $\mathscr{A}_n$, an $n$-stage model for $A$ in normal form. Write $\mathscr{A}_n = \mathscr{A}_{n-1} \otimes_{d_A} \Lambda_n(W^n)$, and construct a subspace $V^n \subset W^n$ using the models of the attaching maps, as explained at the beginning of Section 2.*

*Then, the formality of $X$ is equivalent to the following conditions on $\mathscr{A}_n$ and $V^n$:*

(I) *The composition*

$$W^n \xrightarrow{[d_A]} H^{n+1}\mathscr{A}_{n-1} \to H^{n+1}\mathscr{A}_{n-1}/DH^{n+1}\mathscr{A}_{n-1}$$

*is onto, when restricted to $V^n$.*

(II) *There exists a d.g.a. map $\varrho_{n-1} : \mathscr{A}_{n-1} \to H^*\mathscr{A}_{n-1}$ with $H^p\varrho_{n-1} = \mathrm{id}$ for $p \le n$, and such that $H^{n+1}\varrho_{n-1}$ leaves the subspace $[d_A](V^n)$ invariant.*

**Proof.** We show first that the formality of $X$ implies condition (II), condition (I) being immediate, with Lemmas 1.8, 3.1 and 3.3. We extend $\mathscr{X}_n$ to a model of $X$, say $\mathscr{X}$, and we have, by formality, a d.g.a. map $\psi : \mathscr{X} \to H^*\mathscr{X}$, such that $\psi^* = \mathrm{id}$. We consider the restrictions of $\psi$, denoted by

$$\mathscr{X}_n \xrightarrow{\psi_n} H^*\mathscr{X}_n \quad \text{and} \quad \mathscr{A}_{n-1} \xrightarrow{\varrho_{n-1}} H^*\mathscr{A}_{n-1}.$$

Denoting by $K$ the inclusion $\mathscr{A}_{n-1} \hookrightarrow \mathscr{X}_n$, we note that we have $K^*\varrho_{n-1} = \psi_n K$, and deduce that $H^{n+1}\varrho_{n-1}$ leaves $\operatorname{Ker} H^{n+1}K$ invariant. Looking at the cohomology exact sequence of the elementary extension $K : \mathscr{A}_{n-1} \hookrightarrow \mathscr{X}_n$, we see that

$$\operatorname{Ker} H^{n+1}K = [d_A](V^n).$$

Conversely, let (I) and (II) be satisfied. Using Lemma 1.6, it will be sufficient to construct a d.g.a. map $\psi_n : \mathscr{X}_n \to H^*\mathscr{X}_n$, such that $H^p\psi_n = \mathrm{id}$ for $p \le n+1$. We denote by $\psi_{n-1} : \mathscr{A}_{n-1} \to H^*\mathscr{X}_n$, the composition $K^*\varrho_{n-1}$. Condition (II) says that $H^{n+1}\psi_{n-1} \circ [d_X] = 0$, therefore there exists an extension of $\psi_{n-1}$ over $\mathscr{X}_n$, say $\psi_n'$. We can modify $\psi_n' \mid V^n$ and obtain a new extension, $\psi_n$, which in addition satisfies $\psi_n^* \mid [C^n_{\mathscr{X}_n}] = \mathrm{id}$. Condition (II) also shows $A$ to be formal, hence, by Lemma 3.3, condition (I) implies that $H^*X$ is spherically generated, therefore $H^p\mathscr{X}_n$ is generated by $[C_{\mathscr{T}_n}]$, for $p \le n+1$; by construction $\psi_n^* | [C_{\mathscr{T}_n}^p] = \mathrm{id}$, for any $p$, which shows $H^p \psi_n = \mathrm{id}$ for $p \le n+1$, as asserted.

**3.5. Remark.** Let $X$ be a finite complex, cohomologically simply connected. Theorem 3.4, combined with the construction described in the previous section (see Corollary 2.5), provides a finite test for the formality of $X$. We point out that one can produce simple examples which show that the conditions (I) and (II) in Theorem 3.4 cannot be relaxed. Unfortunately, the examples also show that condition (II) depends on the choice of $\varrho_{n-1}$.

**3.6. Corollary.** *Let $X$ be a finite complex, cohomologically simply connected. Consider a model of its cellular decomposition, inductively constructed as in Corollary 2.5. If we have*

$$W_n^n = W_{n+1}^n + C_n^n \quad (\textit{up to } \dim X - 1),$$

*then $X$ is formal.*

**Proof.** In order to apply Theorem 3.4, we will put $X^n = A$, $X^{n+1} = X$, and inductively suppose that $A$ is formal. Working with the noations used in Theorem 3.4, we write the exact sequence

$$W^n \xrightarrow{[d_A]} H^{n+1}\mathscr{A}_{n-1} \to H^{n+1}\mathscr{A}_n,$$

we deduce that $[d_A]$ is onto, and we observe that the hypothesis $W^n = V^n + C_{\mathscr{A}_n}^n$ implies that $\mathrm{Im}[d_A] = \mathrm{Im}[d_X]$, hence $\langle d_X \rangle$ is onto, which proves condition (I).

The formality of $A$ implies the existence of a d.g.a. map $\varrho_{n-1} : \mathscr{A}_{n-1} \to H^*\mathscr{A}_{n-1}$, such that $H^p \varrho_{n-1} = \mathrm{id}$ for $p \le n$. It is now clear that (II) is also satisfied, since any such map leaves $[d_A](V^n) = H^{n+1}\mathscr{A}_{n-1}$ invariant.

**3.7. Remark.** The algebraic condition $W_n^n = W_{n+1}^n + C_n^n$ corresponds to requiring the attaching map $\vee f_\lambda : \vee S^n \to X^n$ to be formal.

Instead of imposing conditions on the attaching maps, in order to obtain formality, we will now fix $A$, a formal space, cohomologically simply connected and with finitely dimensional rational homology, and an integer $n$, $n \ge$ c-dim $A$. We will give two kinds of conditions on $A$, related to the question of the formality of the spaces $X$, obtained by finitely attaching $(n+1)$-cells to $A$.

**3.8. Lemma.** *Let $A, X$, be as above.*

(i) *If $DH^{n+1}\mathscr{A}_{n-1} = H^{n+1}\mathscr{A}_{n-1}$, then any subspace $V^n \subset W^n$ satisfies conditions* (I) *and* (II) *in Theorem* 3.4. *Therefore any adjunction space $X$ is formal.*

(ii) *If $DH^{n+1}\mathscr{A}_{n-1} = 0$ and $C_{\mathscr{A}_n}^n = 0$, then the only subspace $V^n \subset W^n$ which satisfies conditions* (I) *and* (II) *is $V^n = W^n$.*

*Therefore the only adjunction spaces $X$ which are formal are those which have* *the rational homotopy type of the connected sum of $A$ with a finite bouquet of spheres:* $A\vee(\vee S^{n+1})$.

**Proof.** (i) Condition (I) is trivially verified. The formality of $A$ produces $\varrho_{n-1}:\mathscr{A}_{n-1}\to H^*\mathscr{A}_{n-1}$, such that $H^p\varrho_{n-1}=\mathrm{id}$ for $p\leq n$. Since $H^{n+1}\mathscr{A}_{n-1}=DH^{n+1}\mathscr{A}_{n-1}$, it follows that $H^{n+1}\varrho_{n-1}=\mathrm{id}$, which proves condition (II).

(ii) The assumptions made give an isomorphism

$$W^n \xrightarrow[\sim]{\langle d_A\rangle} H^{n+1}\mathscr{A}_{n-1}=H^{n+1}\mathscr{A}_{n-1}/DH^{n+1}\mathscr{A}_{n-1}.$$

If $V^n$ satisfies condition (I), it follows that $V^n=W^n$.

Lemma 3.8(ii) suggests that the formality condition is in general a very restrictive one, in the following precise sense:

**3.9. Proposition.** *If $A$ is a formal space, cohomologically simply connected and with finite-dimensional rational homology, then there exists an integer $n_A$, $n_A\geq$ c-dim $A$, such that for any $n\geq n_A$, the only spaces $X$, obtained from $A$ by finitely attaching $(n+1)$-cells, which are formal, are those of the rational homotopy type of a connected sum:* $A\vee(\vee S^{n+1})$.

**Proof.** Let $\mathscr{A}$ be a model for $A$, in normal form. It clearly suffices to show that, for $n\geq n_A$, the conditions (ii) in Lemma 3.8 are satisfied. We will denote by $c$ the number c-dim$\,A$, and by $g$ the maximum positive dimension $s$ for which $H^sA/DH^sA\neq 0$. We then take $n_A=c+g$. The formality of $A$ implies that $H^*\mathscr{A}$ is spherically generated (see Lemma 1.8). This in turn is used to prove in a standard way that the natural projection $C_{\mathscr{A}}\to H^+\mathscr{A}/DH^+\mathscr{A}$, is an isomorphism. We deduce that, for $n>g$, we have $C^n_{\mathscr{A}_n}=C^n_{\mathscr{A}}=0$. In order to finish the proof, we suppose $n\geq n_A$, and show $DH^{n+1}\mathscr{A}_{n-1}=0$. Start with $\alpha\in DH^{n+1}\mathscr{A}_{n-1}$. $H^*\mathscr{A}$ being spherically generated, the image of $\alpha$ in $H^{n+1}\mathscr{A}$ will equal the cohomology class of an element of the form $a=\sum z_jb_j$, where $z_j\in C_{\mathscr{A}}$ and $b_j\in\Lambda^+(C_{\mathscr{A}})$. With a little more care, we can even write $\alpha=\sum [z_j][b_j]$ in $H^*\mathscr{A}_{n-1}$. Since $\deg z_j\leq g$ for any $j$, we have that $\deg b_j>$ c-dim $\mathscr{A}$ for any $j$, hence $[b_j]=0$ in $H^*\mathscr{A}_{n-1}$ for any $j$, which finally shows $\alpha=0$.

## 4. Decompositions with minimum number of cells

Let $X$ be a complex, simply connected and of finite type. Examining the exact homotopy sequence of the pair $(X^{n+1},X^n)$, one can easily deduce the inequality

$$\mathrm{rk}\,\pi_{n+1}(X^{n+1},X^n)\geq \mathrm{rk}\,\pi_n(X^n)-\mathrm{rk}\,\pi_n(X^{n+1}).$$

Starting from this simple observation, we were led to make the following definition:

**4.1. Definition.** If $X$ is a complex, denote by $k$ the smallest positive integer $s$ with the property that the $s$-skeleton is not reduced to a single point. The complex $X$ is *economically structured* if, for any $n \geq k$, we have

$$\text{number of } (n+1)\text{-cells} = \operatorname{rk} \pi_n(X^n) - \operatorname{rk} \pi_n(X^{n+1}). \tag{4.2}$$

For a fixed $n$, condition (4.2) says that, in passing from $X^n$ to $X$, the rank of $\pi_n(X)$ is realized without waste of $(n+1)$-cells. From this point of view, the complexes we introduced should be considered as the simplest cellular structures, namely those for which, ignoring torsion, the system of the homotopy groups is constructed with a minimum number of cells.

The use of the results of Section 2 gives the following characterization:

**4.3. Proposition.** *$X$ is economically structured iff we have*

$$C_{\mathscr{X}_n} = C^k_{\mathscr{X}_n} \quad \textit{for any } n,$$

*where $\mathscr{X}_n$ is any model of $X^n$, in normal form, and $k$ is as in the previous definition.*

**Proof.** The first non-trivial skeleton is $X^k = \vee S^k$. We start with $\mathscr{X}_{k,k} = \Lambda_k(W^k_k)$, with zero differential, which represents a $k$-stage model in normal form of $X^k$. We use construction (2.4) to obtain, for any $n \geq k$, an $n$-stage model in normal form of $X^n$, denoted by $\mathscr{X}_{n,n}$. We then extend these models to models in normal form: $\mathscr{X}_n = \Lambda(W_n)$. We observe now that the condition in the proposition trivially holds for $n < k$, and that (4.2) is equivalent, for $n \geq k$, to $r_{n+1} = \dim W^n_n - \dim W^n_{n+1}$, in (2.4) notations Proposition 2.3 shows that $X$ is economically structured iff $C^{n+1}_{n+1} = 0$ for $n \geq k$. After these preliminaries, we will inductively prove the direct implication, which is the non-trivial one. Since c-dim $X^k \leq k$, we deduce from construction (1.1) that $C^p_k = 0$ for $p > k$; the fact that $C^p_k = 0$ for $p < k$, immediately follows, by looking at $\mathscr{X}_{k,k}$. Suppose $C_n = C^k_n$ for $n \geq k$. By a similar c-dim argument, we have $C^p_{n+1} = 0$ for $p > n+1$, and also for $p = n+1$, by hypothesis. By construction, $C^n_{n+1} \subset C^n_n$ and $C^p_{n+1} = C^p_n$ for $p < n$ (see (2.4)), which closes the induction, and the proof.

We are naturally led to introduce the following property, which depends only on the rational homotopy type:

**4.4. Definition.** Let $X$ be a space, cohomologically simply connected and with finite type rational homology. We will say that $X$ is *economic* (as a rational homotopy type) if its rational homotopy type contains an economically structured complex.

This property has the following simple characterization:

**4.5. Theorem.** *Let $X$ be a space, cohomologically simply connected and with finite-*

*dimensional rational homology. $X$ is economic iff there exists an integer $k$ such that $C_{\mathcal{M}} = C^k_{\mathcal{M}}$, $\mathcal{M}$ being any model of $X$, in normal form.*

**Proof.** For the direct implication, it visibly suffices to prove the claim for $X$ economically structured. Choose $k$ as in Proposition 4.3. For any $n$ we will have, according to Corollary 2.5, $C^n_{\mathcal{M}} = C^n_{\mathcal{F}_{n+1}}$, and the conclusion follows from Proposition 4.3. For the other implication, it will be sufficient to prove the following assertion: if an integer $n \geq k$ is chosen, such that c-dim $\mathcal{M} \leq n$, then there exists a finite complex, $(k-1)$-connected, at most $n$-dimensional, and economically structured which has $\mathcal{M}$ as a minimal model. The assertion is a replica of Proposition 2.7, and the proof also goes parallelly, with the following changes: the induction starts with $n = k$, and we can take $X = \vee S^k$. For the inductive step, we observe that, if $C_{\mathcal{M}} = C^k_{\mathcal{M}}$, then $C_{\mathcal{A}} = C^k_{\mathcal{A}}$, and that the complex $X$, constructed as in Proposition 2.7, will be economically structured iff $r = \operatorname{cod} V^n$; since by that construction, $r$-cod $V^n = \dim C^{n+1}_{\mathcal{M}}$, and since $n \geq k$, the hypothesis in our theorem helps to finish the proof.

**4.6. Definition.** Let $X$ be a space. $H^*X$ *is homogenously generated* if there exists an integer $l$ with the property that $H^lX$ generates $H^*X$ as an algebra.

The characterization in Theorem 4.5 takes a particularly simple form, when a formality-like assumption is added:

**4.7. Corollary.** *Let $X$ be as in Theorem* 4.5 *and suppose further that $H^*X$ is spherically generated. Then $X$ is economic iff $H^*X$ is homogeneously generated.*

**Proof.** We notice the following general facts, omitting the proofs, since they are standard: if $\mathcal{M}$ is a minimal algebra, and if $H^*\mathcal{M}$ is generated by $H^l\mathcal{M}$, then $C_{\mathcal{M}} = C^l_{\mathcal{M}}$, and $H^*\mathcal{M}$ is spherically generated. On the other hand, if $C_{\mathcal{M}} = C^l_{\mathcal{M}}$, and $H^*\mathcal{M}$ is spherically generated, $\mathcal{M}$ being in normal form, then $H^l\mathcal{M}$ generates $H^*\mathcal{M}$.

Taking into account the criterion of Theorem 4.5, the corollary follows.

**4.8. Remarks.** The previous proof also shows that the inverse implication is valid without any additional assumptions. On the other hand, there are examples which show that the direct implication no longer holds, if the spherical generation hypothesis on $X$ is dropped.

In the proof of Corollary 4.7 we also noticed that, if $H^*X$ is homogeneously generated, then $H^*X$ is spherically generated. The following example shows that the property of homogeneous generation is not strong enough to ensure the formality. The example also illustrates the utility of the cellular formality criterion given in Section 3.

**4.9. Example.** We will use the notations of Theorem 3.4, and take $A = P^2C \vee P^3C$

and $n=7$. Since $A$ is formal, it will be sufficient to construct a 7-model in normal form: $\alpha_7:\mathscr{A}_7\to H^*A$. $H^*A$ is generated by elements $a^*$ and $b^*$, $\deg a^*=\deg b^*=2$, with the relations $a^{*3}=0$, $b^{*4}=0$, and $a^*b^*=0$. We describe the required 7-stage model, constructed by the algorithm (1.1):

$W^2$: generated by $a$ and $b$, with $da=db=0$, and $\alpha_7 a=a^*$, $\alpha_7 b=b^*$;
$W^3$: generated by $x$, with $dx=ab$, and $\alpha_7 x=0$;
$W^5$: generated by $y$, with $dy=a^3$, and $\alpha_7 y=0$;
$W^6$: generated by $z$, with $dz=by-a^2x$, and $\alpha_7 z=0$;
$W^7$: generated by $u$ and $v$, with $du=az-xy$, $dv=b^4$, and $\alpha_7 u=\alpha_7 v=0$.

Taking as subspace $V^7$, the subspace spanned by $u+v$, and $r=\operatorname{cod} V^7=1$, we denote by $X$ the space obtained by attaching one 8-cell to $A$, which realizes these algebraic data (see Proposition 2.6). We remark that $A$ is economically structured, therefore, by construction, $X$ will be economically structured. We will now show that $V^7$ satisfies condition (I) of Theorem 3.4, but not condition (II). It will follow, from Lemma 3.3, that $H^*X$ is spherically generated, hence, by Corollary 4.7, $H^*X$ is homogenously generated. By Theorem 3.4, $X$ is not formal.

Since $H^8\mathscr{A}_6$ is spanned by the classes $[du]$ and $[dv]$, and $DH^8\mathscr{A}_6$ is spanned by $[dv]$, condition (I) immediately follows. The d.g.a. maps $\varrho_6:\mathscr{A}_6\to H^*\mathscr{A}_6$, with the property $H^p\varrho_6=\mathrm{id}$ for $p\le 7$, are given by $\varrho_6 a=[a]$, $\varrho_6 b=[b]$, $\varrho_6 x=0$, $\varrho_6 y=0$, and $\varrho_6 z=m[b^3]$, with a parameter $m\in Q$. It is equally immediate to see that, for any value of $m, H^8\varrho_6$ does not leave $[d_A](V^7)$ invariant.

**4.10. Remarks.** By contrast with the previous example one can still infer the formality from additional hypotheses.

Let $H^*X$ be generated by $H^l$, and with cup-length $c$. If $l\ge c$, then $X$ is formal (using the obstructions to formality developed in [5]). If $l\ge c-1$ and $H^*X$ satisfies Poincaré duality, then $X$ is formal (using [8]). For details, see [9].

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